Eigenstates of the Lewis-Riesenfeld invariant for de Sitter and quasi-de Sitter universes

Research-oriented Project

Presented by
Max Joseph Fahn
July 20, 2018

Institute for Theoretical Physics
Chair for Theoretical Physics III (Quantum Gravity)
Friedrich-Alexander-Universität Erlangen-Nürnberg

The results of this research project were published in
Max Joseph Fahn, Kristina Giesel, and Michael Kobler. "Dynamical Properties of the Mukhanov-Sasaki Hamiltonian in the context of adiabatic vacua and the Lewis-Riesenfeld invariant." Universe 5.7 (2019): 170.

Supervisors:

Prof. Dr. Kristina Giesel
Michael Kobler, M.Sc.

Abstract

We investigate the Mukhanov-Sasaki equation in the context of inflationary cosmology. In spatially flat gauge, this equation describes the dynamics of linear perturbations in the inflaton field and it can be brought into the form of a time-dependent harmonic oscillator equation. After quantising a single mode of this equation, ladder operators are contructed and along with them the spectrum of the associated quantum mechanical operator is constructed. From this, an expression for the ground and first excited state of the eigenfunctions of the Lewis-Riesenfeld invariant, an invariant of the time-dependent harmonic oscillator, is derived. To be able to write down this expression explicitly, it is necessary to first find a solution for the Mukhanov-Sasaki equation and after that use it to construct a solution for the Ermakov equation. The latter appears indirectly in the explicit expression of the eigenstates of the Lewis-Riesenfeld invariant. For the background universe a quasi-de Sitter spacetime, which is almost exponential expansion, is considered.

Acknowledgements

I want to thank **Prof. Dr. Kristina Giesel** for giving me the opportunity to work on this great project and in her group. I highly appreciate the fantastic lectures on cosmology, observables and further interesting topics she hold for her group.

I also want to thank **Michael Kobler** for all the excellent information he provided me with concerning the project and the entire topic, for the fruitful discussions about my results and for the patience with me.

Contents

1 Introduction to modern cosmology

Mankind has always been curious. No matter which country or culture, human beings have always been looking to the sky and wondered, what was out there and whether this universe was eternal. Today we know that the universe is evolving and we can measure these dynamics with a high degree of precision. Together with the evidence of this dynamics comes the question of how everything started at the beginning and what happened to make the universe look like it does nowadays. Cosmology tries to give an answer to these questions.
In this field of physics models of the very early universe are built and improved based on observations of the cosmos today. Einstein's theory of general relativity delivers the framework to formulate cosmological models. This theory is based on the fundamental principle of diffeomorphism invariance, that is the covariance of all fundamental equations with respect to the choice (and change) of coordinates and therefore the invariance of the action under diffeomorphisms. At the core of the theory of general relativity are the Einstein equations, which connect matter in the universe with the geometry of spacetime and give us, compared to Newton's laws, a new interpretation for gravity: Gravity is not a force generated by masses, but a result of energies and momenta of particles changing the geometry of spacetime. In abstract index notation and in Planck units Einstein's equations read:

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R + \Lambda\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu} \tag{1}$$

with gravitational constant G, energy-momentum tensor $T_{\mu\nu}$ that contains information about the content of the universe, cosmological constant Λ, as well as metric tensor $g_{\mu\nu}$, Ricci tensor $R_{\mu\nu}$ and Ricci scalar R, which all encode information about the geometry and curvature of spacetime. Finding the general solution to these equations is complicated and, indeed, is impossible in full generality at least at the analytical level. Only under certain conditions, like the assumption of specific symmetries, is it possible to solve them. In cosmology, one tries to achieve this and derives physical properties about the different epochs in the evolution of our universe from the Einstein equations.
A first simplification is provided by the cosmological principle, which postulates spatial homogeneity and isotropy on sufficiently large scales. Homogeneity means that spatial slices look the same at every point, so shifts in position do not change anything; isotropy means that space looks the same in any direction, so rotations leave physical quantities invariant. An experimental indication that supports this principle is the cosmic microwave background radiation (CMB), which one can measure across the sky with nearly the same temperature of ≈ 2.7 K. Deviations from this temperature are of the order 10^{-4} to 10^{-5} K (see e.g. [1]).
Considering the (possible) spatial curvature of the universe, there are three types of maximally symmetric spaces that can be constructed. The first one with positive curvature

is isomorphic to the 3-sphere S^3, the second one with negative curvature is isomorphic to the 3-hyperboloid H^3 and the third one with zero curvature is isomorphic to a flat Euclidean space E^3. Requiring maximal spatial symmetry, it turns out that only one degree of freedom for the metric tensor remains. This is the scale factor $a(t)$ of the spacelike hypersurfaces that gives information about the spatial size of the universe at time t. These spacetimes with maximally symmetric spatial submanifolds are called Friedmann-Lemaître-Robertson-Walker (FLRW) universes and often used in cosmology. For calculations, it is usually more convenient to work with conformal time η instead of cosmological time t. The former is related to cosmological time by

$$\eta(t) = \int_{t_i}^{t} \frac{dt}{a(t)} \ . \tag{2}$$

Here, t_i denotes initial time, so the time of the Big Bang. Investigating the universe right after the Big Bang and modelling its content as a perfect fluid, there are several types of matter that could have dominated in that epoch. For different prevailing matter types, the dynamical equations, which are the so called Friedmann equations that one obtains from the simplified Einstein equations that respect the cosmological symmetries, lead to different solutions for the scale factor. An interesting type is the de Sitter universe which is a dark energy dominated universe - this dark energy is a possible interpretation for the cosmological constant Λ in the Einstein equations (1). The two Friedmann equations involving a cosmological constant are given by:

$$\frac{1}{a^2(t)}\left(\frac{da}{dt}(t)\right)^2 = \frac{8\pi}{3}\rho(t) - \frac{k}{a^2(t)} + \frac{\Lambda}{3} \tag{3}$$

and

$$\frac{1}{a(t)}\frac{d^2a}{dt}(t) = -\frac{4\pi}{3}(\rho(t) + 3p(t)) + \frac{\Lambda}{3} \tag{4}$$

with curvature parameter k, energy density ρ and pressure p of the matter under consideration. Solving these equations for the scale factor $a(t)$ leads to an exponential expansion of the universe in the de Sitter case. Today we expect an epoch of exponential expansion right after the Big Bang (a reason for that is given in the next section), so a de Sitter universe is a good candidate to describe the processes that took place there. A particular mechanism there is considered to solve several existing problems in cosmology.

One of these issues, the flatness problem, is the fact that today we measure a nearly flat spatial universe even though the change of curvature since the Big Bang has always been strictly positive according to the Friedmann equations, if the universe is dominated by standard matter. This is matter with an equation of state parameter $w > -\frac{1}{3}$, where w describes the relation between pressure p and density ρ of the considered matter by $p = w\rho$. According to that definition, radiation or cosmic dust are standard matter while dark energy is not (for derivations see for instance [2]). To achieve such a situation with a nearly flat universe today but a strictly increasing curvature since the Big Bang, an extreme fine-tuning of the initial conditions of the universe would be necessary, which is considered very unnatural in physics and therefore undesirable. A possible solution to this and further problems (see introductory books on cosmology, for example [3]) is provided by the mechanism of inflation. This dynamically generates a period of accelerated expansion in the very early universe. At a sufficiently long duration (measured in terms of e-foldings of the scale factor) it is able to solve a series of problems in non-inflationary cosmology.

2 Theory prerequisites

In this chapter we introduce the basic theory needed to follow the developments of the project. First we discuss inflation, perturbation theory and gauge invariance in the cosmological framework. In this context we find the Mukhanov-Sasaki equation, which is the equation of motion for each Fourier mode of the gauge invariant Mukhanov-Sasaki variable. Further we show that this equation can be rewritten as the differential equation of a harmonic oscillator with a certain time-dependent frequency. After deriving an expression for this frequency for a de Sitter and quasi-de Sitter background, we introduce the Lewis-Riesenfeld invariant, which is a dynamical invariant of the time-dependent harmonic oscillator. Finally, we use a Bogoliubov transformation to map each mode of the time-dependent Mukhanov-Sasaki equation to a time-independent one.

2.1 The Mukhanov-Sasaki equation

2.1.1 From inflation to the Mukhanov-Sasaki equation

The simplest possible choice for inflation is to introduce a real scalar field ϕ called inflaton. Requiring homogeneity, isotropy and assuming a minimal coupling of the inflaton field to gravity with a potential energy density $V(\phi)$ gives an action functional for the inflaton, from which one can derive its equation of motion on an FLRW spacetime:

$$\ddot{\phi} + 3H(t)\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \tag{5}$$

with the Hubble function

$$H(t) := \frac{\dot{a}(t)}{a(t)} \ . \tag{6}$$

In this equation of motion appears a term proportional to $\dot{\phi}$ which is a friction term often called Hubble friction since it would be absent for a flat spacetime. Requiring inflation to have lasted long enough in order to solve the issues mentioned above implies that $\ddot{\phi}$ is subdominant in the equation of motion, which constitutes the so-called slow-roll approximation, leading to

$$3H(t)\dot{\phi} + \frac{\partial V}{\partial \phi} \approx 0 \tag{7}$$

and also to

$$H(t) \approx \text{const} \tag{8}$$

during inflation, which corresponds to approximately exponential expansion of the universe (see (6)). Therefore, the de Sitter solution is a suitable background solution during inflation. We extend this background as a next step with perturbations. We introduce such perturbations (for a general FLRW solution) to adopt the model of the universe to reality because not everything in the universe is distributed homogeneously and isotropically as is shown by our everyday experience. Even on larger scales there are deviations from the cosmological principle, the CMB, ineed, shows fluctuations of the order 10^{-4} to 10^{-5} K. As these are several orders of magnitude below the mean value, it is still reasonable to treat the background as a FLRW universe and add small perturbations (up to first order) which break the spatial symmetries. These perturbations are introduced for the geometry of spacetime as well as for the matter degrees of freedom whose energy density and pressure now is not exactly the same everywhere. Together with these perturbations comes the problem of choosing an appropriate coordinate system. While in the FLRW spacetime there was a convenient one, namely the one of a comoving observer, now in one coordinate system a certain quantity may appear free of perturbations, in another it may not. Physical quantities must not depend on the choice of the coordinate system, so we are only interested in quantities that are unaltered by coordinate transformations. One of them is given by the Mukhanov-Sasaki variable $v(\eta, \vec{x})$, for more details we refer the rader to [4]. In spatially flat gauge, the gauge where one can treat degrees of freedom in matter and geometry independently at the linear order to a good approximation, the expression for the Mukhanov-Sasaki variable is

$$v(\eta, \vec{x}) = a(\eta)\ \delta\phi(\eta, \vec{x})\ . \tag{9}$$

In this gauge it describes the linear inflaton perturbation $\delta\phi(\eta, \vec{x})$ on spatially flat slices. By expanding the inflaton action in terms of $v(\eta, \vec{x})$ up to second order and performing a Fourier transform, one can derive the equation of motion for each mode $\vec{k}$ of the Fourier transformed Mukhanov-Sasaki variable. This equation of motion is called the Mukhanov-Sasaki equation and it reads in conformal time without a specific gauge-fixing:

$$\left(\frac{\partial^2}{\partial\eta^2} + k_a k^a - z^{-1}(\eta)\frac{\partial^2 z}{\partial\eta^2}(\eta)\right) v_{\vec{k}}(\eta) = 0\ , \tag{10}$$

with

$$z^2(\eta, \vec{x}) := \frac{a^2(\eta)}{\mathcal{H}^2(\eta)}\left(\frac{\partial\phi}{\partial\eta}\right)^2 \tag{11}$$

or expressed in cosmological time

$$z^2(t, \vec{x}) := \frac{a^2(t)}{H^2(t)}\left(\frac{\partial\phi}{\partial t}\right)^2\ . \tag{12}$$

In equation (11), $\mathcal{H}(\eta)$ denotes the Hubble function with respect to conformal time, which is $\mathcal{H}(\eta) = \frac{a'(\eta)}{a(\eta)}$ corresponding to $a(t)H(t)$, as $a' = \frac{da}{d\eta} = \frac{dt}{d\eta}\frac{da}{dt} = a\dot{a}$ with a prime denoting the derivative with respect to η and a dot denoting the derivative with respect to t. In the last step we used the definition of the conformal time in (2). $z(\eta, \vec{x})$ (see (11)) has an interesting form: It consists of a mixture of geometrical degrees of freedom (in $a(\eta)$) and matter degrees of freedom (in ϕ) of the background, and does not depend on the choice of time.

Introducing an explicitly time-dependent frequency

$$\omega_{\vec{k}}^2(\eta) := k_a k^a - z^{-1}(\eta)\frac{\partial^2 z}{\partial \eta^2}(\eta) \tag{13}$$

into equation (10), this equation can be rewritten as:

$$\left(\frac{\partial^2}{\partial \eta^2} + \omega_{\vec{k}}^2(\eta)\right) v_{\vec{k}}(\eta) = 0 \ , \tag{14}$$

which is exactly the differential equation of a harmonic oscillator with explicitly time-dependent frequency for every mode $\vec{k}$! Before we look for solutions to this equation, first we want to explicitly calculate this Mukhanov-Sasaki frequency (13) for a de Sitter and quasi-de Sitter background spacetime.

2.1.2 Explicit expression for a de Sitter and quasi-de Sitter background

Our main goal is to find the frequency $\omega_{\vec{k}}(\eta)$ and thus to calculate $\frac{z''(\eta)}{z(\eta)}$. Considering the slow-roll approximation, we can introduce a (first) slow-roll parameter

$$\epsilon := -\frac{\dot{H}(t)}{H^2(t)} \ll 1 \ . \tag{15}$$

This parameter measures the fractional change of $H(t)$ per Hubble time and is much smaller than one in order to make the possible changes in $H(t)$ small such that inflation lasts long enough. Using the two Friedmann equations together with the energy density and pressure of the inflaton we can rewrite ϵ as

$$\epsilon = \frac{4\pi}{H^2(t)}\left(\frac{d\phi}{dt}\right)^2 \ . \tag{16}$$

With the latter, we get for $z(\eta)$

$$z^2(\eta) = \frac{a^2(\eta)\epsilon}{4\pi} \ . \tag{17}$$

Starting with the first derivative of $z(\eta)$ divided by $z(\eta)$ expressed in cosmological time

$$\frac{z'(\eta)}{z(\eta)} = \mathcal{H}(\eta) + \frac{1}{2}\frac{\epsilon'}{\epsilon} \to a(t)H(t) + \frac{1}{2}\frac{a(t)\dot{\epsilon}}{\epsilon} =: a(t)H(t)\left(1 + \frac{1}{2}\tau\right) \ , \tag{18}$$

we can define a second slow-roll parameter measuring the fractional change of ϵ per Hubble time

$$\tau := \frac{\dot{\epsilon}}{H(t)\epsilon} \ll 1 \ . \tag{19}$$

The requirement that this parameter is much smaller than one is equal to the requirement of slowly varying ϵ which makes it reasonable to consider ϵ as almost constant. The derivative with respect to cosmological time t of equation (18) yields

$$\frac{d}{dt}\left(\frac{z'(\eta)}{z(\eta)}\right) = \ddot{a}(t)\left(1+\frac{\tau}{2}\right) + a(t)H^2(t)\frac{\tau\kappa}{2} \ , \tag{20}$$

where we introduced a third and final slow-roll parameter according to

$$\kappa := \frac{\dot{\tau}}{H(t)\tau} \ll 1 \ . \tag{21}$$

Then using (15) with $\dot{H}(t)$ we calculated the resulting expression

$$\epsilon = 1 - \frac{\ddot{a}(t)}{a(t)\ H^2(t)} \ , \tag{22}$$

that is equivalent to

$$\ddot{a}(t) = a(t)\ H^2(t)\ (1-\epsilon) \ . \tag{23}$$

Inserting this into (20) we obtain

$$\begin{aligned}\frac{z''(\eta)}{z(\eta)} = \frac{d}{d\eta}\left(\frac{z'(\eta)}{z(\eta)}\right) + \left(\frac{z'(\eta)}{z(\eta)}\right)^2 &= \frac{dt}{d\eta}\left(\frac{z'(\eta)}{z(\eta)}\right)^{\cdot} + \left(\frac{z'(\eta)}{z(\eta)}\right)^2 = \\ &= (a(t)H(t))^2\left(2-\epsilon+\frac{3\tau}{2}+\frac{\tau\kappa}{2}-\frac{\epsilon\tau}{2}+\frac{1}{4}\tau^2\right)\end{aligned} \tag{24}$$

Since we are interested in a first order equation (in fact, $v(\eta,\vec{x})$ is a linearized gauge invariant variable) it suffices to truncate (24) after all first order terms, which yields

$$\frac{z''(\eta)}{z(\eta)} \approx (a(t)H(t))^2\left(2-\epsilon+\frac{3}{2}\tau\right) \ . \tag{25}$$

From (22) and (15), which state that $\forall\eta$ we have $\epsilon \ll 1$, so $\epsilon - 1 \approx$ *const*, we get the equation

$$-\frac{\ddot{a}(t)}{a(t)\ H^2(t)} = \frac{d}{d\eta}\left(\frac{1}{a(t)\ H(t)}\right) = \epsilon - 1 \approx \text{const} \ , \tag{26}$$

from which we find by integration with respect to η:

$$-\frac{1}{\eta}\left(\frac{1}{a(t)\ H(t)}\right) = 1-\epsilon \ . \tag{27}$$

Expanding this for $a(t)H(t)$ in a Taylor series, where we only take terms up to first order in ϵ, yields:

$$a(t)H(t) \approx -\frac{1}{\eta}(1+\epsilon) \ . \tag{28}$$

Substituting this into equation (25) and dropping all terms of higher than first order in the slow-roll parameters, we obtain

$$\frac{z''(\eta)}{z(\eta)} \approx \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right) \tag{29}$$

with

$$\nu := \frac{3}{2} + \epsilon + \frac{\tau}{2} \ . \tag{30}$$

This is equivalent to (25) if one drops all terms of order two in the slow-roll parameters in ν^2. So the frequency on quasi-de Sitter reads finally:

$$\omega_{\vec{k}}^2(\eta) = \vec{k}^2 - \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right) \ . \tag{31}$$

To obtain the frequency on de Sitter, we set all slow-roll parameters equal to zero and find:

$$\omega_{\vec{k}}^2(\eta) = \vec{k}^2 - \frac{2}{\eta^2} \ . \tag{32}$$

2.2 The Lewis-Riesenfeld invariant and its connection to the time-independent harmonic oscillator via a Bogoliubov transformation

As discussed before, the Mukhanov-Sasaki equation (14) has the same form as the equation of motion for an explicitly time-dependent harmonic oscillator with frequency $\omega_{\vec{k}}(\eta)$ for every mode $\vec{k}$. In this section first we want to introduce the Lewis-Riesenfeld invariant I_{LR}, which is a dynamical invariant of the time-dependent harmonic oscillator under a certain condition. As far as the project in this work is considered, we will directly discuss the quantised operator $\hat{I}_{LR}$ corresponding to the classical invariant I_{LR}. More information on the quantisation can be found for example in [2], section 4. The explicit expression of this invariant as in [5] and [2] is given by

$$\hat{I}_{LR}(\eta) := \frac{1}{2}\left(\left(\xi(\eta)\,\hat{p} - \xi'(\eta)\,\hat{q}\right)^2 + \frac{\omega_0^2\,\hat{q}^2}{\xi^2(\eta)} \right) . \tag{33}$$

Since $\hat{I}_{LR}$ is an invariant it has to satisfy

$$\frac{d\hat{I}_{LR}}{d\eta} = \frac{1}{i\hbar}[\hat{I}_{LR}(\eta), \hat{H}(\eta)] + \frac{\partial \hat{I}_{LR}}{\partial \eta} = 0 \tag{34}$$

for the Hamiltonian of a time-dependent harmonic oscillator

$$\hat{H}(\eta) := \frac{\hat{p}^2}{2} + \frac{1}{2}\,\omega^2(\eta)\,\hat{q}^2 . \tag{35}$$

This carries over to a condition on $\xi(\eta)$ that has to fulfill the non-linear differential equation

$$\left(\frac{d^2}{d\eta^2} + \omega^2(\eta)\right)\xi(\eta) - \omega_0^2\,\xi(\eta)^{-3} = 0 . \tag{36}$$

This equation is called the Ermakov equation, see for instance [6] for further information. For a time-independent oscillator

$$\hat{H}_0 := \frac{\hat{p}^2}{2} + \frac{1}{2}\omega_0^2\,\hat{q}^2 \tag{37}$$

with $\xi(\eta) = const$, $\hat{I}_{LR}$ becomes a rescaled harmonic oscillator

$$\hat{I}_{LR} = \frac{\xi^2\,\hat{p}^2}{2} + \frac{1}{2\,\xi^2}\omega_0^2\,\hat{q}^2 . \tag{38}$$

From equation (34) then follows $\xi^2 = 1$ in this case. Reinserting this back into the Ermakov equation (36) yields

$$\omega^2(\eta) = \omega_0^2 \tag{39}$$

as a consistency check.

As the next step, we want to investigate the ground state and first excited state of the spectrum of $\hat{I}_{LR}$. Therefore, we introduce the following notation for the n-th eigenvalue and -vector in the eigenvalue equation for $\hat{I}_{LR}$:

$$\hat{I}_{LR}\,|\lambda_n,\eta\rangle = \lambda_n\,|\lambda_n,\eta\rangle\ . \tag{40}$$

To construct these states, first we introduce a transformation that maps each mode of the time-dependent Mukhanov-Sasaki Hamiltonian into a time-independent one. This transformation is a time-dependent Bogoliubov transformation, which is a unitary transformation conserving the commutator algebra of that operators related to the harmonic oscillator before and after the transformation, $[\hat{a},\hat{a}^\dagger] = [\hat{a}(\eta),\hat{a}^\dagger(\eta)] = 1$. Starting classically and defining the corresponding canonical transformation, in [2] the author develops a quantised analogue via a generator, that represents a unitary Bogoliubov transformation. In this reference, different representations of this transformation are given as well as a Baker-Campbell-Hausdorff decomposition for concrete applications. The transformation, denoted as $\hat{\Gamma}_\xi$ and that is explicitly time-dependent via $\xi(\eta)$, is unitary for $\xi \in \mathbb{R}$, that is

$$\hat{\Gamma}_\xi^\dagger\,\hat{\Gamma}_\xi = \hat{\Gamma}_\xi\,\hat{\Gamma}_\xi^\dagger = \hat{I}\ , \tag{41}$$

acts on states $|i\rangle \in \mathscr{H}$ in a Hilbert space $\mathscr{H}$ by $\hat{\Gamma}_\xi\,|i\rangle$ and on operators $\hat{O} \in \mathrm{End}(\mathscr{H})$ by $\hat{\Gamma}_\xi^\dagger\,\hat{O}\,\hat{\Gamma}_\xi$. An explicit expression for $\hat{\Gamma}_\xi$ is given by

$$\hat{\Gamma}_\xi := exp\left[\bar{\alpha}(\xi)\,\hat{\sigma}_- - \alpha(\xi)\,\hat{\sigma}_+ + i\mu(\xi)\,\hat{\sigma}_3\right] \tag{42}$$

with $\hat{\sigma}_+ := \frac{1}{2}\hat{a}^\dagger\hat{a}^\dagger$, $\hat{\sigma}_- := \hat{\sigma}_+^\dagger$, $\hat{\sigma}_3 := \hat{a}^\dagger\hat{a} + \frac{\hat{I}}{2}$. $\hat{a}^{(\dagger)}$ are the ladder operators corresponding to the time-independent harmonic oscillator, and $\alpha(\xi)$ and $\mu(\xi)$ are two functions that depend on $\xi(\eta)$. For many applications it is convenient to rewrite the transformation into a form consisting of a product of three exponentials, where each exponent is proportional to one of the three sigma operators. This form can be obtained by a generalized Baker-Campbell-Hausdorff decomposition. The standard formalism, $e^{X+Y} = e^X e^Y e^{-\frac{1}{2}[X,Y]}$, does not work in our case, because the conditions required to derive the standard formula, $[X,[X,Y]] = 0$ and $[Y,[Y,X]] = 0$, do not hold. In [2] an alternative path to the BCH-decomposition is presented.

It turns out that this Bogoliubov transformation connects the eigenstates and ladder operators of the time-independent harmonic oscillator ($|n\rangle$ and $\hat{a}^{(\dagger)}$) to the ones of the Lewis-Riesenfeld invariant ($|\lambda_n,\eta\rangle$ and $\hat{a}^{(\dagger)}(\eta)$) in the following way:

$$\hat{\Gamma}_\xi^\dagger\,|n\rangle = |\lambda_n,\eta\rangle \tag{43}$$

and

$$\hat{\Gamma}_\xi^\dagger\,\hat{a}^{(\dagger)}\,\hat{\Gamma}_\xi = \hat{a}^{(\dagger)}(\eta)\ . \tag{44}$$

This is justified by the application of the transformation on the n-th eigenstate of the time-independent harmonic oscillator,

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n |0\rangle \ , \tag{45}$$

that is

$$\hat{\Gamma}_\xi^\dagger |n\rangle = \hat{\Gamma}_\xi^\dagger \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n \hat{\Gamma}_\xi \hat{\Gamma}_\xi^\dagger |0\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger(\eta)\right)^n |\lambda_0, \eta\rangle =: |\lambda_n, \eta\rangle \ , \tag{46}$$

where we used the unitarity of the transformation several times in the second step for

$$\hat{\Gamma}_\xi^\dagger \left(\hat{a}^\dagger\right)^n \hat{\Gamma}_\xi = \hat{\Gamma}_\xi^\dagger \, \hat{a}^\dagger \, \hat{\Gamma}_\xi \, \hat{\Gamma}_\xi^\dagger \, \hat{a}^\dagger \, \hat{\Gamma}_\xi \, ... \, \hat{\Gamma}_\xi^\dagger \, \hat{a}^\dagger \, \hat{\Gamma}_\xi = \left(\hat{\Gamma}_\xi^\dagger \, \hat{a}^\dagger \, \hat{\Gamma}_\xi\right)^n =: \left(\hat{a}^\dagger(\eta)\right)^n \ . \tag{47}$$

The ladder operators associated with $\hat{I}_{LR}$ are given by

$$\hat{a}(\eta) := \sqrt{\frac{\omega_0}{2\hbar}} \left[\frac{i}{\omega_0} \left(\xi(\eta)\hat{p} - \xi'(\eta)\hat{q}\right) + \frac{\hat{q}}{\xi(\eta)}\right] \tag{48}$$

and

$$\hat{a}^\dagger(\eta) := \sqrt{\frac{\omega_0}{2\hbar}} \left[-\frac{i}{\omega_0} \left(\xi(\eta)\hat{p} - \xi'(\eta)\hat{q}\right) + \frac{\hat{q}}{\xi(\eta)}\right] \ . \tag{49}$$

We find as expected

$$\hat{I}_{LR}(\eta) = \hbar\omega_0 \left(\hat{a}^\dagger(\eta) \, \hat{a}(\eta) + \frac{1}{2}\right) \tag{50}$$

and

$$[\hat{a}(\eta), \hat{a}^\dagger(\eta)] = 1 \ , \tag{51}$$

which holds for every η. From these identities it is obvious that

$$\hat{I}_{LR}(\eta) = \hbar\omega_0 \left(\hat{a}^\dagger(\eta) \, \hat{a}(\eta) + \frac{1}{2}\right) = \hbar\omega_0 \left(\hat{\Gamma}_\xi^\dagger \, \hat{a}^\dagger \, \hat{\Gamma}_\xi \, \hat{\Gamma}_\xi^\dagger \, \hat{a} \, \hat{\Gamma}_\xi + \frac{1}{2}\right) = \hat{\Gamma}_\xi^\dagger \left[\hbar\omega_0 \left(\hat{a}^\dagger \, \hat{a} + \frac{1}{2}\right)\right] \hat{\Gamma}_\xi \tag{52}$$

where we used the unitarity of the transformation. According to the definition, this is

$$\hat{I}_{LR}(\eta) = \hat{\Gamma}_\xi^\dagger \, \hat{H}_0 \, \hat{\Gamma}_\xi \ . \tag{53}$$

For the time-independent case, where we found $\xi(\eta) = 1$, $\alpha(\xi) = \mu(\xi) = 0$, the Bogoliubov transformation becomes the identity transformation and yields

$$\hat{I}_{LR} = \hat{H}_0 \ . \tag{54}$$

Using this Bogoliubov transformation, we will derive an expression for the lowest two eigenstates of $\hat{I}_{LR}$ in the position representation in section 3.1.

3 Eigenstates of the Lewis-Riesenfeld invariant for (quasi-)de Sitter universes

In this section we present the core calculations and results of the project. We start by computing the spectrum of the Lewis-Riesenfeld invariant in position representation. As we use the Bogoliubov transformation from section 2 to perform this, our result contains $\xi(\eta)$ which is obtained by solving the Ermakov equation for a certain spacetime background. We choose to work on a quasi-de Sitter spacetime. To construct this solution of the Ermakov equation, first we need to solve the Mukhanov-Sasaki equation for quasi-de Sitter. It turns out that the solution is related to Bessel functions. With this solution, we find an expression for $\xi(\eta)$. In the last part, we reinsert this into the ground and first excited state eigenfunctions of the Lewis-Riesenfeld invariant, and plot and analyse these states.

3.1 The spectrum of the Lewis-Riesenfeld invariant in position representation

3.1.1 Ground state

Starting from the defining equation for the harmonic oscillator vacuum,

$$\hat{a}\,|0\rangle = 0\ , \tag{55}$$

due to the linearity of the operator $\hat{\Gamma}^\dagger_\xi$ the following equation is valid:

$$\hat{\Gamma}^\dagger_\xi\,\hat{a}\,|0\rangle = 0\ . \tag{56}$$

Using the unitarity of $\hat{\Gamma}_\xi$, we can insert an identity in order to obtain

$$\hat{\Gamma}^\dagger_\xi\,\hat{a}\,\hat{\Gamma}_\xi\,\hat{\Gamma}^\dagger_\xi\ |0\rangle = \hat{a}(\eta)\,|\lambda_0,\eta\rangle = 0\ , \tag{57}$$

which, in terms of positions and momenta, corresponds to

$$\left[\frac{i}{\omega_0}\left(\xi\hat{p} - \xi'\hat{q}\right) + \frac{\hat{q}}{\xi}\right]|\lambda_0,\eta\rangle = 0\ . \tag{58}$$

As a next step we use the position representation and get

$$\left[\frac{1}{\omega_0}\left(\xi\hbar\frac{\partial}{\partial q}\right) - q\left(\frac{i}{\omega_0}\xi' - \frac{1}{\xi}\right)\right]\psi_0(q,\eta) = 0\ , \tag{59}$$

where we defined the ground state in the position basis

$$\psi_0(q,\eta) := \langle q|\lambda_0,\eta\rangle\ . \tag{60}$$

Equation (59) is a first order differential equation for $\psi_0(q,\eta)$ which has the following solution:

$$\psi_0(q,\eta) = C(\eta)\, e^{\left(i\frac{\xi'}{2\hbar\xi} - \frac{\omega_0}{2\hbar\xi^2}\right)q^2} \tag{61}$$

with a normalisation constant $C(\eta)$. This constant can be determined by requiring conservation of probability for any time, that is

$$\int_{-\infty}^{\infty} dq\, |\psi_0(q,\eta)|^2 = 1 \ . \tag{62}$$

This yields upon integration

$$C(\eta) = \sqrt[4]{\frac{\omega_0}{\pi\hbar\, \xi^2(\eta)}} \cdot e^{i\theta} \ . \tag{63}$$

Without loss of generality, we can set the phase θ to zero and get the ground state wave function of the eigenstate of the Lewis-Riesenfeld invariant in position representation:

$$\psi_0(q,\eta) = \sqrt[4]{\frac{\omega_0}{\pi\hbar\xi^2}}\, e^{\left(i\frac{\xi'}{2\hbar\xi} - \frac{\omega_0}{2\hbar\xi^2}\right)q^2} = \sqrt[4]{\frac{\omega_0}{\pi\hbar\xi^2}}\, e^{-\frac{\omega_0}{2\hbar\xi^2}q^2} \left[\cos\left(\frac{\xi'}{2\hbar\xi}q^2\right) + i\sin\left(\frac{\xi'}{2\hbar\xi}q^2\right)\right] \ . \tag{64}$$

In this expression, ξ and ξ' encode the dynamics of the background spacetime under consideration.

3.1.2 First excited state

Starting with (61), we now want to find the first excited state

$$|\lambda_1,\eta\rangle = \hat{a}_\xi^\dagger\, |\lambda_0,\eta\rangle \ . \tag{65}$$

This calculation is straightforward, we start by plugging in the explicit form of the raising operator which gives an expression for $|\lambda_1,\eta\rangle$:

$$|\lambda_1,\eta\rangle = \left[-\frac{i}{\omega_0}\left(\xi\hat{p} - \xi'\hat{q}\right) + \frac{\hat{q}}{\xi}\right] |\lambda_0,\eta\rangle \ . \tag{66}$$

Next, considering again the position representation we get:

$$\begin{aligned}\langle q|\lambda_1,\eta\rangle =: \psi_1(q,\eta) &= \sqrt[4]{\frac{\omega_0}{\pi\hbar\xi^2}}\sqrt{\frac{\omega_0}{2\hbar}}\left[-\frac{\hbar\xi}{\omega_0}\frac{\partial}{\partial q} + \left(\frac{1}{\xi} + \frac{i\xi'}{\omega_0}\right)q\right] e^{\left(i\frac{\xi'}{2\hbar\xi} - \frac{\omega_0}{2\hbar\xi^2}\right)q^2} = \\ &= \sqrt[4]{\frac{\omega_0}{\pi\hbar\xi^2}}\sqrt{\frac{\omega_0}{2\hbar}}\,\frac{2}{\xi}\, q\, e^{\left(i\frac{\xi'}{2\hbar\xi} - \frac{\omega_0}{2\hbar\xi^2}\right)q^2} \ .\end{aligned} \tag{67}$$

By definition of the ladder operators, this state is already properly normalised. This can be checked also explicitly:

$$\int_{-\infty}^{\infty} dq \, |\psi_1(q,\eta)|^2 = \sqrt{\frac{\omega_0}{\pi\hbar\xi^2}} \, \frac{\omega_0}{2\hbar} \, \frac{4}{\xi^2} \int_{-\infty}^{\infty} dq \, q^2 e^{-\frac{\omega_0}{\hbar\xi^2} q^2} = \sqrt{\frac{\omega_0}{\pi\hbar\xi^2}} \, \frac{\omega_0}{2\hbar} \, \frac{4}{\xi^2} \, \frac{\sqrt{\pi}}{2} \left(\frac{\hbar\xi^2}{\omega_0}\right)^{\frac{3}{2}} = 1. \tag{68}$$

So we found the first excited eigenstate of $\hat{I}_{LR}$ in position representation:

$$\psi_1(q,\eta) = \sqrt[4]{\frac{\omega_0}{\pi\hbar\xi^2}} \, \sqrt{\frac{\omega_0}{2\hbar}} \, \frac{2}{\xi} \, q \, e^{\left(i\frac{\xi'}{2\hbar\xi} - \frac{\omega_0}{2\hbar\xi^2}\right) q^2} \; . \tag{69}$$

The higher excited states can be calculated analoguously by

$$|\lambda_n, \eta\rangle = \frac{1}{\sqrt{n!}} \, \left(\hat{a}_\xi^\dagger\right)^n |\lambda_0, \eta\rangle \; . \tag{70}$$

In order to write down the full expression for the ground or first excited state in position representation we need to find an explicit form of $\xi(\eta)$. This function will be derived in the next two sections for a quasi-de Sitter universe. For a de Sitter universe this was already achieved in [2].

3.2 General solution of the Mukhanov-Sasaki equation for quasi-de Sitter

Using the previously derived result (31) for the frequency of the time-dependent harmonic oscillator Mukhanov-Sasaki equation (14) on quasi-de Sitter, the full equation reads:

$$v''_{\vec{k}}(\eta) + \left(\vec{k}^2 - \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right)\right) v_{\vec{k}}(\eta) = 0 \ , \tag{71}$$

which is equivalent to:

$$\eta^2 v''_{\vec{k}}(\eta) + \left(\vec{k}^2\eta^2 - \left(\nu^2 - \frac{1}{4}\right)\right) v_{\vec{k}}(\eta) = 0 \ . \tag{72}$$

As $\vec{k}$ only appears quadratic in this equation and $\vec{k}^2 = ||\vec{k}||^2$, we reduce the dependency of $v_{\vec{k}}(\eta)$ to a dependency on the absolute value of $\vec{k}$, which we define as $k := ||\vec{k}||$. The results can be generalized to higher-dimensional wave vectors $\vec{k}$ straightforwardly. Our next aim is to construct a general solution for this equation for every mode k. In order to find these solutions, we try to cast the Mukhanov-Sasaki equation into the form of a generalized Bessel differential equation. Therefore, we perform several steps, starting with a substitution:

$$v_k(\eta) := \sqrt{\eta} w_k(\eta) \ . \tag{73}$$

The first and second derivatives of $v_k(\eta)$ with respect to η then are:

$$v'_k(\eta) = \frac{1}{2\sqrt{\eta}} w_k(\eta) + \sqrt{\eta} w'_k(\eta) \tag{74}$$

$$v''_k(\eta) = \frac{1}{\sqrt{\eta}} w'_k(\eta) + \sqrt{\eta} w''_k(\eta) + \frac{1}{4\sqrt{\eta^3}} w_k(\eta) \ . \tag{75}$$

Plugging this into equation (72) and dividing by $\sqrt{\eta}$ (as we are only interested in the inflationary period $\eta \in (-\infty, 0)$ and therefore $\eta \neq 0$), we get:

$$\eta^2 w''_k(\eta) + \eta w'_k(\eta) + \left(k^2\eta^2 - \nu^2\right) w_k(\eta) = 0 \ . \tag{76}$$

Secondly, we introduce another substitution given by:

$$\chi = -k\eta \tag{77}$$

and get

$$w_k(\eta) = w_k\left(-\frac{\chi}{k}\right) =: \tilde{w}_k(\chi) \tag{78}$$

$$w'_k(\eta) = \frac{dw_k}{d\eta} = \frac{d\tilde{w}_k}{d\eta} = \frac{d\tilde{w}_k}{d\chi}\frac{d\chi}{d\eta} = -\frac{d\tilde{w}_k}{d\chi} k =: -\tilde{w}'_k(\chi)\, k \ , \tag{79}$$

where the prime here denotes the derivative with respect to the argument. In the end, the substituted equation now looks like the following:

$$\chi^2 \tilde{w}_k''(\chi) + \chi \tilde{w}_k'(\chi) + (\chi^2 - \nu^2)\tilde{w}_k(\chi) = 0 \ . \tag{80}$$

This is the generalized Bessel differential equation with $\nu \in \mathbb{R}$. Solutions to this equation are well-known, for more detailled information we refer the reader for example to [7], [8] or [9]. We use two different complete solutions, firstly a linear combination of Bessel functions J_ν and Neumann functions N_ν, and secondly a linear combination of Hankel functions $H_\nu^{(1/2)}$ of the first and second kind. We will work with both solutions individually, what gives us more flexibility when calculating asymptotics and also provides us with a consistency check in the end. These two solutions are given by

$$\tilde{w}_{k,1}(\chi) = \tilde{a}_1 J_\nu(\chi) + \tilde{b}_1 \, N_\nu(\chi) \tag{81}$$

$$\tilde{w}_{k,2}(\chi) = \tilde{a}_2 H_\nu^{(1)}(\chi) + \tilde{b}_2 \, H_\nu^{(2)}(\chi) \ , \tag{82}$$

and solve equation (80) due to its linearity. Using those, a solution for the Mukhanov-Sasaki equation in (71) is then of the form:

$$v_{k,1}(\eta) = C_1 \sqrt{\eta} \left(\tilde{a}_1 J_\nu(-k\eta) + \tilde{b}_1 \, N_\nu(-k\eta) \right) \tag{83}$$

and the second linearly independent solution reads:

$$v_{k,2}(\eta) = C_2 \sqrt{\eta} \left(\tilde{a}_2 H_\nu^{(1)}(-k\eta) + \tilde{b}_2 \, H_\nu^{(2)}(-k\eta) \right) \ , \tag{84}$$

where C_1 and C_2 are two up to now arbitrary normalisation constants. Each of the two solutions forms a general solution for every individual mode k of the Mukhanov-Sasaki equation (71) by itself.

3.3 Constructing a solution to the Ermakov equation on quasi-de Sitter

As a next step, we want to construct a solution to the Ermakov equation on quasi-de Sitter. Therefore, we will use the procedure outlined in [2], in appendix C. Starting with two linearly independent solutions $u_k(\eta)$ and $v_k(\eta)$ that solve (71), a solution for the Ermakov equation can be constructed as:

$$\xi_k(\eta) := \sqrt{A_k\, u_k(\eta)^2 + 2B_k\, u_k(\eta)v_k(\eta) + C_k\, v_k(\eta)^2} \tag{85}$$

along with the following condition on the involved constants:

$$A_k C_k - B_k^2 = k^2\, W^{-2}(u_k(\eta), v_k(\eta)) \ , \tag{86}$$

where the Wronskian determinant is defined as:

$$W(u_k(\eta), v_k(\eta)) := u_k(\eta)v_k'(\eta) - u_k'(\eta)v_k(\eta) \ . \tag{87}$$

In order to find a solution, first we have to investigate the Wronskian determinant of the Bessel functions of different kinds that will appear in our derivation. This will be done in the next section. Furthermore, we expect the Wronskian determinant of the complete solutions to be constant, since we have

$$\frac{dW}{d\eta}(u_k(\eta),(v_k(\eta)) = u_k(\eta)v_k''(\eta) - v_k(\eta)u_k''(\eta) =: W'(u_k(\eta), v_k(\eta)) \ , \tag{88}$$

where $u_k(\eta)$ and $v_k(\eta)$ are both solutions of the Mukhanov-Sasaki equation (71), that is

$$\eta^2 u_k''(\eta) + \left(k^2\eta^2 - \left(\nu^2 - \frac{1}{4}\right)\right) u_k(\eta) = 0 \tag{89}$$

$$\eta^2 v_k''(\eta) + \left(k^2\eta^2 - \left(\nu^2 - \frac{1}{4}\right)\right) v_k(\eta) = 0 \ . \tag{90}$$

Multiplying the first equation with $v_k''(\eta)$ and subtracting from it the second one multiplied by $u_k''(\eta)$ yields an expression involving the previously derived Wronskian:

$$\left(k^2\eta^2 - \left(\nu^2 - \frac{1}{4}\right)\right) (u_k(\eta)v_k''(\eta) - v_k(\eta)u_k''(\eta)) = 0 \ , \tag{91}$$

which is equivalent to

$$\left(k^2\eta^2 - \left(\nu^2 - \frac{1}{4}\right)\right) W'(u_k(\eta), v_k(\eta)) = 0 \ . \tag{92}$$

This equation has to hold for any η, therefore the derivative of the Wronskian determinant has to vanish for any η, showing that the Wronskian determinant itself is a constant if its arguments obey a harmonic equation of motion.

3.3.1 Wronskian determinant of Bessel, Neumann and Hankel functions

We want to evaluate

$$W(H_\nu^{(1)}(z), H_\nu^{(2)}(z)) = H_\nu^{(1)}(z) H_\nu^{(2)\prime}(z) - H_\nu^{(1)\prime}(z) H_\nu^{(2)}(z) \ . \tag{93}$$

Considering the form of the generalized Bessel differential equation,

$$y''(z) + \frac{1}{z} y'(z) + \left(1 - \frac{\nu^2}{z^2}\right) y(z) = 0 \ , \tag{94}$$

this can be brought into a different form according to

$$y''(z) + p(z)\, y'(z) + q(z)\, y(z) = 0 \ , \tag{95}$$

with appropriate choices for the rational functions p and q and we assume that we have two general solutions, $y_1(z)$ and $y_2(z)$. The Wronskian determinant of these two solutions is given by:

$$W(y_1(z), y_2(z)) = y_1(z)\, y_2'(z) - y_2(z)\, y_1'(z) \tag{96}$$

and its derivative with respect to z reads

$$W'(y_1(z), y_2(z)) = y_1(z)\, y_2''(z) - y_2(z)\, y_1''(z) \ . \tag{97}$$

Expressing the second derivatives with the help of equation (95), we find

$$\begin{aligned} W'(y_1(z), y_2(z)) &= y_1(z)\,(-p(z)\, y_2'(z) - q(z)\, y_2(z)) + y_2(z)\,(p(z)\, y_1'(z) + q(z)\, y_1(z)) = \\ &= -p(z)\,(y_1(z)\, y_2'(z) - y_2(z)\, y_1'(z)) = -p(z)\, W(z) \ , \end{aligned} \tag{98}$$

where $p(z) = \frac{1}{z}$ follows from (95). That gives the following differential equation for the Wronskian determinant of the Hankel functions:

$$W'(H_\nu^{(1)}(z), H_\nu^{(2)}(z)) = -\frac{1}{z}\, W(H_\nu^{(1)}(z), H_\nu^{(2)}(z)) \ . \tag{99}$$

The solution for this reads

$$W(H_\nu^{(1)}(z), H_\nu^{(2)}(z)) = \frac{c}{z} \tag{100}$$

with an arbitrary constant c considering the fact that $W'(H_\nu^{(1)}(z), H_\nu^{(2)}(z))$ is constant. To determine this constant, we consider the case of very large z, where the Hankel functions can be expressed in an asymptotic form as

$$\lim_{z\to\infty} H^{(1/2)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{\pm i(z+a)} \tag{101}$$

with $a := \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}$. Evaluating the Wronskian determinant in this limit gives:

$$\lim_{z\to\infty} W(H_\nu^{(1)}(z), H_\nu^{(2)}(z)) = -\frac{1}{\pi z^2} - \frac{2i}{\pi z} + \frac{1}{\pi z^2} - \frac{2i}{\pi z} = -\frac{4i}{\pi z} \ , \tag{102}$$

where we used the smoothness of $H^{(1/2)}$ on $(0,\infty)$ to switch the order of the limit and the derivation, as well as the following equality:

$$\lim_{z\to\infty} H^{(1/2)\prime}(z) = -\frac{1}{\sqrt{2\pi z^3}}\, e^{\pm i(z+a)} \pm i\sqrt{\frac{2}{\pi z}}\, e^{\pm i(z+a)} \ . \tag{103}$$

So $c = -\frac{4i}{\pi}$ and therefore it needs to hold that

$$W(H_\nu^{(1)}(z), H_\nu^{(2)}(z)) = -\frac{4i}{\pi z} \ . \tag{104}$$

Using the definition of the Hankel functions, $H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z)$ and $H_\nu^{(2)}(z) = J_\nu(z) - iN_\nu(z)$ respectively, we also find a final expression for $W(N_\nu(z), J_\nu(z))$:

$$\begin{aligned} W(H_\nu^{(1)}(z), H_\nu^{(2)}(z)) = \\ = (J_\nu(z) + iN_\nu(z))(J'_\nu(z) - iN'_\nu(z)) - (J_\nu(z) - iN_\nu(z))(J'_\nu(z) + iN'_\nu(z)) = \\ = 2i(J'_\nu(z)N_\nu(z) - J_\nu(z)N'_\nu(z)) = 2iW(N_\nu(z), J_\nu(z)) \end{aligned} \tag{105}$$

and thus we conclude:

$$W(N_\nu(z), J_\nu(z)) = -\frac{2}{\pi z} \ . \tag{106}$$

3.3.2 Solution to the Ermakov equation using Hankel functions

In equation (84) we found a general solution of the Mukhanov-Sasaki equation in terms of Hankel functions. Both summands of this solution fulfill the Mukhanov-Sasaki equation individually. To construct a solution for the Ermakov equation, according to equation (85) we need two linear independent solutions $u_k(\eta)$ and $v_k(\eta)$ of the Mukhanov-Sasaki equation. We can find such solutions in the two summands in (84) by setting the prefactors C_2, $\tilde{a}_2$ and $\tilde{b}_2$ unequal to zero. For convenience we set them equal to one and define:

$$u_k(\eta) := \sqrt{\eta}\, H_\nu^{(1)}(-k\eta) \ , \tag{107}$$

$$v_k(\eta) := \sqrt{\eta}\, H_\nu^{(2)}(-k\eta) \ . \tag{108}$$

In order to determine the coefficients A_k, B_k and C_k in (85), first of all we require a well-defined limit for $\eta \to -\infty$. We interpret this condition as an initial condition, as we want our solution to fulfill the equation also in this limit. This gives, using the asymptotic formula of the Hankel functions for large $|z|$ (that is the form in (101)):

$$\eta\, H_\nu^{(1)}(-k\eta)\, H_\nu^{(2)}(-k\eta) \longrightarrow -\frac{2}{\pi k} \ , \tag{109}$$

$$\eta \left(H_\nu^{(1/2)}(-k\eta) \right)^2 \longrightarrow -\frac{2}{\pi k}\, e^{\mp 2i\left(k\eta + \left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right)} \ . \tag{110}$$

So, the sum under the square root in (85) becomes in this limit

$$A_k \eta \left(H_\nu^{(1)}(-k\eta) \right)^2 + C_k \eta \left(H_\nu^{(2)}(-k\eta) \right)^2 \tag{111}$$

and is only well-defined for the choice $A_k = C_k = 0$. This gives

$$\xi_k(\eta) = \sqrt{2 B_k \eta\, H_\nu^{(1)}(-k\eta)\, H_\nu^{(2)}(-k\eta)} \ . \tag{112}$$

The value of B_k is fixed by the use of (86) and evaluation of (112) such that $\xi_k(\eta)$ is real. As the condition in (86) is only set on B_k^2, we are free to choose an arbitrary sign for B_k. As we consider the interval $\eta \in (-\infty, 0)$ we choose B_k to be negative to get a real $\xi_k(\eta)$:

$$B_k = -\frac{k}{|W(u_k(\eta), v_k(\eta))|} \ . \tag{113}$$

The Wronskian determinant yields with inserting $u_k(\eta)$ and $v_k(\eta)$:

$$W(u_k(\eta), v_k(\eta)) = W(\sqrt{\eta}\, H_\nu^{(1)}(-k\eta), \sqrt{\eta}\, H_\nu^{(2)}(-k\eta)) = \tag{114}$$

$$= k\eta\, W(H_\nu^{(1)}(-k\eta), H_\nu^{(2)}(-k\eta)) = k\eta \frac{4i}{\pi k\eta} = \frac{4i}{\pi} \ , \tag{115}$$

where we used

$$\frac{d}{d\eta} H_\nu^{(1/2)}(-k\eta) = \frac{d(-k\eta)}{d\eta} \frac{d}{d(-k\eta)} H_\nu^{(1/2)}(-k\eta) =: -H_\nu^{(1/2)\prime}(\chi)\, k \ . \tag{116}$$

This gives

$$B_k = -\frac{\pi}{4} k \tag{117}$$

and therefore the final solution is:

$$\xi_k(\eta) = \sqrt{-\frac{\pi k \eta}{2}\, H_\nu^{(1)}(-k\eta)\, H_\nu^{(2)}(-k\eta)} \ . \tag{118}$$

In the limit $\eta \to -\infty$, that is past infinity in conformal time, we get

$$\lim_{\eta \to -\infty} \xi_k(\eta) = 1 \ , \tag{119}$$

which is consistent with our choices made in section 2.2.

3.3.3 Solution to the Ermakov equation using Bessel and Neumann functions

The procedure here is analoguous to the one in the previous section. We start by defining

$$u_k(\eta) := \sqrt{\eta}\, N_\nu(-k\eta) \ , \tag{120}$$

$$v_k(\eta) := \sqrt{\eta}\, J_\nu(-k\eta) \ . \tag{121}$$

The asymptotic formulas are the real part (for $J_\nu(-k\eta)$) and the imaginary part (for $N_\nu(-k\eta)$) of the first Hankel function (101), so this is proportional to a cosine for the Bessel and a sine for the Neumann function. To get a well-defined past infinity limit for $\xi_k(\eta)$, now B_k has to vanish and we need $A_k = C_k$ to use $A_k \sin^2(x) + A_k \cos^2(x) = A_k$. Together with the relation (86) and the Wronskian determinant for Bessel and Neumann functions in equation (106), we find

$$A_k = C_k = -\frac{\pi}{2} k \ . \tag{122}$$

Therefore we get

$$\xi_k(\eta) = \sqrt{-\frac{\pi k\eta}{2} \left(J_\nu^2(-k\eta) + N_\nu^2(-k\eta)\right)} \ , \tag{123}$$

which also gives a consistent limit for past conformal time:

$$\lim_{\eta \to -\infty} \xi_k(\eta) = 1 \ . \tag{124}$$

This is no surprise, as (123) is equivalent to (118) due to the fact that:

$$H_\nu^{(1)}(z)\, H_\nu^{(2)}(z) = (J_\nu(z) + iN_\nu(z))(J_\nu(z) - iN_\nu(z)) = J_\nu(z)^2 + N_\nu(z)^2 \ . \tag{125}$$

3.3.4 Transition from the quasi-de Sitter to the de Sitter solution of the Ermakov equation for vanishing slow roll parameters

In order to perform a consistency check, we want to see whether we can get the de Sitter solution for the Ermakov equation from the previously found quasi-de Sitter solution in the limit of vanishing slow roll parameters. The solution for de Sitter was derived in [2] and was shown to be

$$\xi_k(\eta) = \sqrt{1 + \frac{1}{(k\eta)^2}} \ . \tag{126}$$

The limit of vanishing slow roll parameters alters ν, which was defined as

$$\nu = \frac{3}{2} + \epsilon + \frac{\tau}{2} \ . \tag{127}$$

For the de Sitter solution, this will become

$$\nu \longrightarrow \frac{3}{2} \ . \tag{128}$$

Starting with the quasi-de Sitter solution (123), we now get the half-integer (of an odd number) Bessel and Neumann functions. In general, they can be represented by (for more details see e.g. [10]):

$$J_{n+\frac{1}{2}}(-k\eta) = \frac{1}{\sqrt{\pi}}(-1)^n(-2k\eta)^{n+\frac{1}{2}}\frac{d^n}{d(k^2\eta^2)^n}\frac{\sin(k\eta)}{k\eta}\ , \tag{129}$$

$$N_{n+\frac{1}{2}}(-k\eta) = (-1)^{n+1}J_{-n-\frac{1}{2}}(-k\eta) = (-1)^{n+1}(-k\eta)^{n+\frac{1}{2}}\sqrt{\frac{2}{\pi}}\frac{d^n}{(-k\eta\ d(-k\eta))^n}\frac{\cos(k\eta)}{-k\eta}\ . \tag{130}$$

In our case $n = 1$ and it therefore holds that:

$$J_{\frac{3}{2}}(-k\eta) = \sqrt{\frac{2}{\pi}}\frac{k\eta\cos(k\eta) - \sin(k\eta)}{(-k\eta)^{\frac{3}{2}}} \tag{131}$$

$$N_{\frac{3}{2}}(-k\eta) = -\sqrt{\frac{2}{\pi}}\frac{k\eta\sin(k\eta) + \cos(k\eta)}{(-k\eta)^{\frac{3}{2}}}\ , \tag{132}$$

where we used for the derivative in $J_{\frac{3}{2}}(-k\eta)$:

$$\frac{d}{dx^2}\frac{\sin(x)}{x} = \frac{d}{dy}\frac{sin(\sqrt{y})}{\sqrt{y}} = \frac{\cos(\sqrt{y}) - \frac{1}{\sqrt{y}}\sin(\sqrt{y})}{2y} = \frac{x\cos(x) - \sin(x)}{2x^3}\ . \tag{133}$$

Reinserting this back into (123) gives the limiting case of $\xi_k(\eta)$ for a de Sitter background:

$$\begin{aligned}\xi_k(\eta) = &\sqrt{\frac{[-k\eta\cos(k\eta) + \sin(k\eta)]^2}{(k\eta)^2} + \frac{[\cos(k\eta) + k\eta\sin(k\eta)]^2}{(k\eta)^2}} = \\ &= \sqrt{\frac{1}{(k\eta)^2}[1 + (k\eta)^2][\sin^2(k\eta) + \cos^2(k\eta)]} = \sqrt{1 + \frac{1}{(k\eta)^2}}\ ,\end{aligned} \tag{134}$$

which is precisely the solution earlier obtained for de Sitter where analoguous conditions on A_k, B_k, C_k and W were imposed as well as the same choices were made regarding the well-definedness of the limit $\eta \to -\infty$.

3.4 Analysis of plots of the ground and first excited state eigenfunctions of the Lewis-Riesenfeld invariant on de Sitter and quasi-de Sitter

In this section we will present and analyse plots of the ground and first excited state of the eigenfunction of I_{LR} on de Sitter and quasi-de Sitter spacetime respectively. For the plots we chose units with $\hbar = \omega_0 = k = 1$.

3.4.1 Ground state

First we discuss the ground state

$$\psi_0(q,\eta) = \sqrt[4]{\frac{\omega_0}{\pi\hbar\xi^2}}\, e^{\left(i\frac{\xi'}{2\hbar\xi} - \frac{\omega_0}{2\hbar\xi^2}\right)q^2} = \sqrt[4]{\frac{\omega_0}{\pi\hbar\xi^2}}\, e^{-\frac{\omega_0}{2\hbar\xi^2}q^2} \left[\cos\left(\frac{\xi'}{2\hbar\xi}q^2\right) + i\sin\left(\frac{\xi'}{2\hbar\xi}q^2\right)\right] , \tag{135}$$

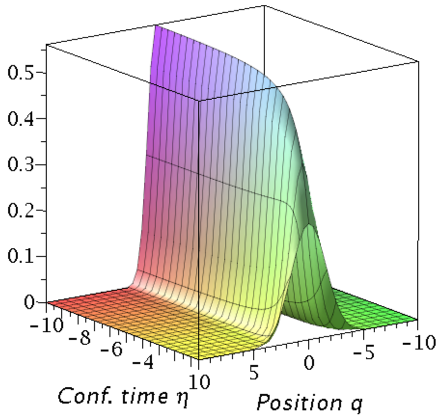
with

$$\xi = \xi(\eta) = \sqrt{-\frac{\pi k\eta}{2}\left(J_\nu^2(-k\eta) + N_\nu^2(-k\eta)\right)} \tag{136}$$

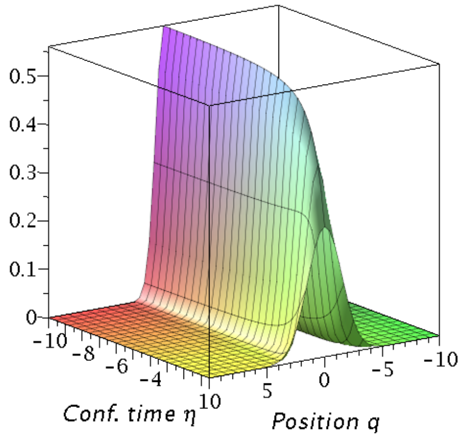
and

$$\xi' = \frac{d\xi}{d\eta} . \tag{137}$$

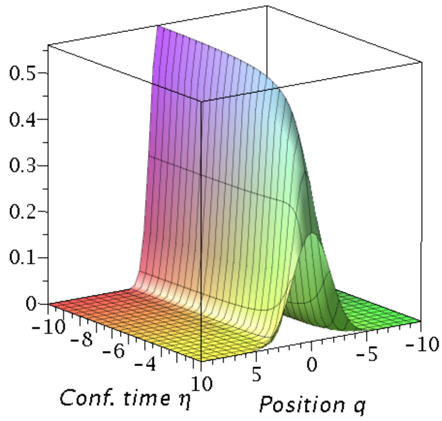
We plotted the modulus square of the ground state in a three-dimensional plot depending on conformal time η and position q in Figure 1. The parameter space was chosen such that the physically interesting region close to zero in conformal time and close to the origin in position becomes apparent. The state has the shape of a Gaussian and is centered around the origin in position. For positions further away from the origin it is very close to zero (except for conformal times close to zero). Evolution in conformal time, which goes from the Big Bang at $\eta = -\infty$ up to $\eta = 0$, leaves the Gaussian almost unchanged except for η close to 0. In this section, the probability distribution spreads in position and decreases at the center. To illustrate this behaviour we plotted snapshots of the distribution for conformal times close to 0 in Figure 2. The last parameter in the formula for the ground state is ν, which determines the order of the Bessel functions and also the deviation of the background spacetime from a de Sitter spacetime, which is characterised by $\nu = 1.5$. Due to the fact that we used approximations around a de Sitter spacetime which gave us the quasi-de Sitter solutions, our formulae are accurate only for values close to the de Sitter solution. Therefore we chose to discuss the quasi-de Sitter cases where $\nu = 1.45$ and $\nu = 1.55$ and compare it to the de Sitter case. All three cases have very similar behaviour. The main difference lies in the spreading of the probability distribution. For $\nu = 1.45$ this spreading happens later (that is for conformal times closer to zero), while for $\nu = 1.55$ the distribution spreads earlier than the de Sitter distribution. This is evident from Figure 2. As the ground state (135) is normalised for arbitrary but fixed η and ν, the different spreading behaviour also implies a difference in the height of the maxima of the distributions. While the maximum for $\nu = 1.45$ is above the de Sitter maximum for η close to zero, the one for $\nu = 1.55$ is below the de Sitter one in that region. So, for a fixed conformal time, the probability distribution for $\nu = 1.45$ has smaller variance and larger maximum than the distribution for de Sitter while the one for $\nu = 1.55$ has larger variance and smaller maximum than de Sitter. For conformal times $\eta \ll 0$ the probability distribution is (almost) the same for a quasi-de Sitter and a de Sitter background spacetime.



(a) Probability density for de Sitter



(b) Probability density for quasi-de Sitter with $\nu = 1.45$



(c) Probability density for quasi-de Sitter with $\nu = 1.55$

Figure 1: Modulus square (probability density) of the ground state eigenfunction of I_{LR} depending on position q and conformal time η.

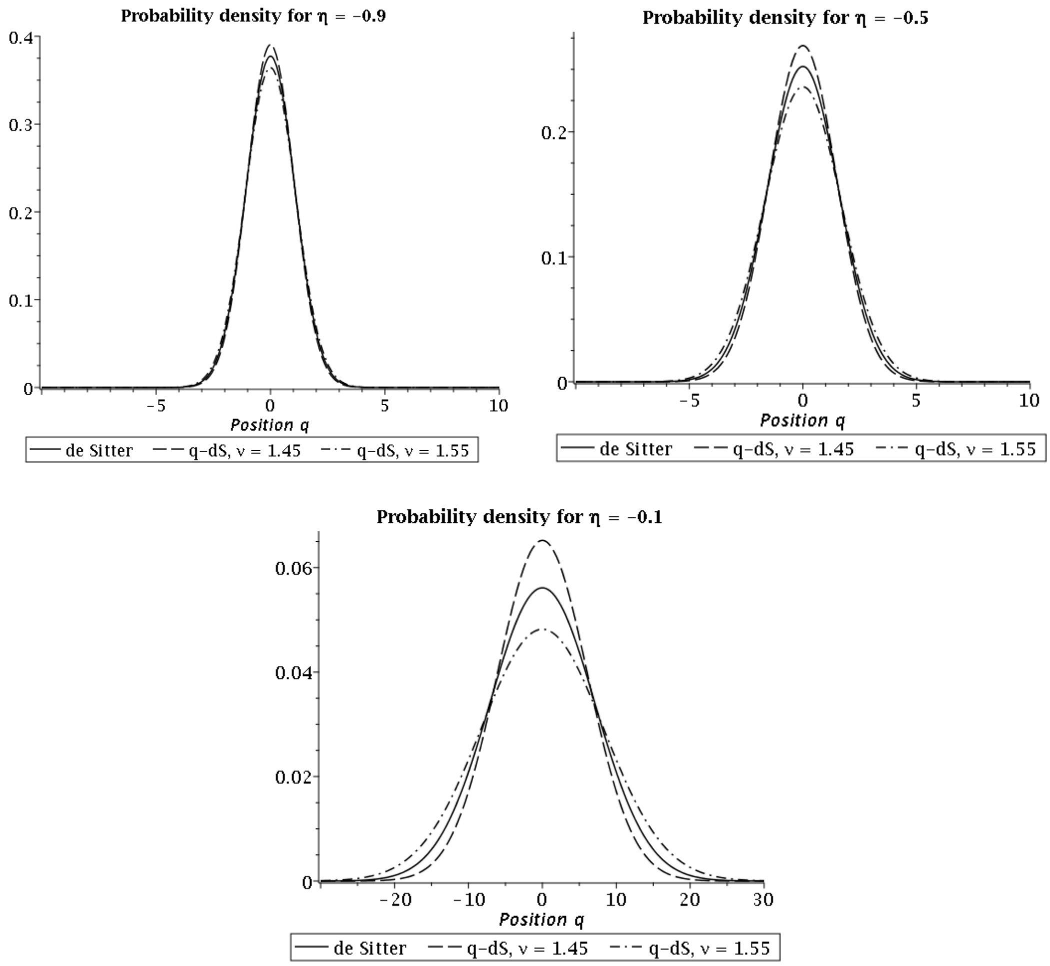


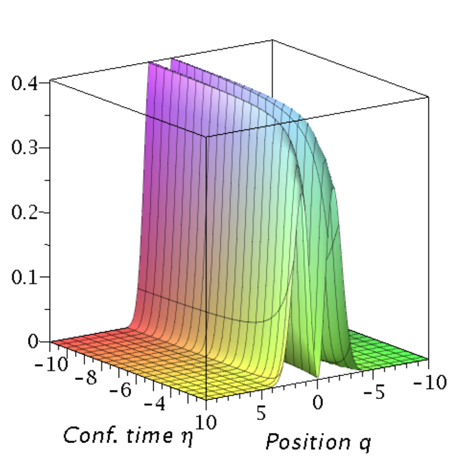
Figure 2: Modulus square of the ground state eigenfunction of I_{LR} depending on position q for fixed conformal times η. The line shapes indicate the background universe (de Sitter or quasi-de Sitter)

3.4.2 First excited state

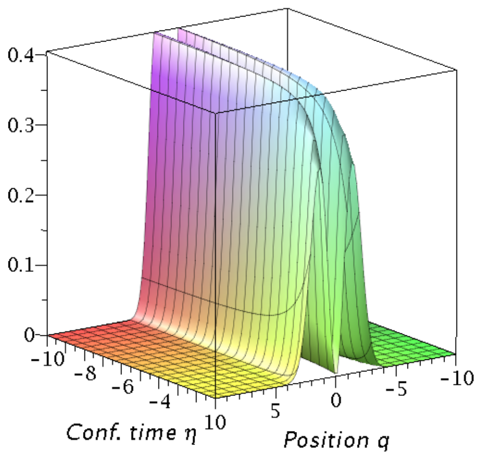
As a second step, we want to discuss plots of the first excited state, which is given by

$$\psi_1(q,\eta) = \sqrt[4]{\frac{\omega_0}{\pi\hbar\xi^2}}\,\sqrt{\frac{\omega_0}{2\hbar}}\,\frac{2}{\xi}\,q\,e^{\left(i\frac{\xi'}{2\hbar\xi}-\frac{\omega_0}{2\hbar\xi^2}\right)q^2} \tag{138}$$

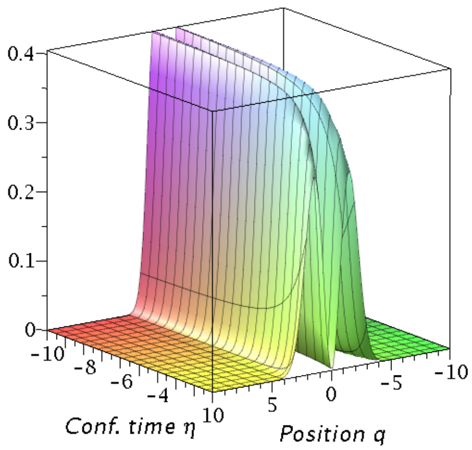
with ξ according to (136) and ξ' according to (137). For the plots we chose the same parameters as we did for the ground state in section 3.4.1. The three-dimensional plot of the modulus square of $\psi_1(q,\eta)$ depending on conformal time η and position q is visualised in Figure 3. Here is apparent that this state has zero probability to be found at the origin in position. At both sides of this origin there is one maximum. As $|\psi_1(q,\eta)|^2$ only depends on q^2 this state is symmetric and thus the maxima have the same height. The behaviour of the state's probability distribution when evolving it in (conformal) time is similar to how the ground state evolved. It remains (almost) constant for smaller η and spreads in position for η close to zero. Due to its normalisation for arbitrary but fixed η and ν we find that the simultaneous decrease of the maxima comes along with a spread of the distribution. This can also be realised in Figure 4, where the probability density for fixed conformal times was plotted. To investigate the deviation of the state from the one for a de Sitter background spacetime, we again considered the cases with $\nu = 1.45$ and $\nu = 1.55$. This deviation has the same influence on the first excited state as it has on the ground state: For $\nu = 1.45$ the spreading happens later and therefore the maxima in this case are higher than the maxima for the de Sitter case. In contrary to that, for $\nu = 1.55$ the distribution spreads earlier and thus has lower maxima than the de Sitter distribution.



(a) Probability density for de Sitter



(b) Probability density for quasi-de Sitter with $\nu = 1.45$



(c) Probability density for quasi-de Sitter with $\nu = 1.55$

Figure 3: Modulus square (probability density) of the first excited state eigenfunction of I_{LR} depending on position q and conformal time η.

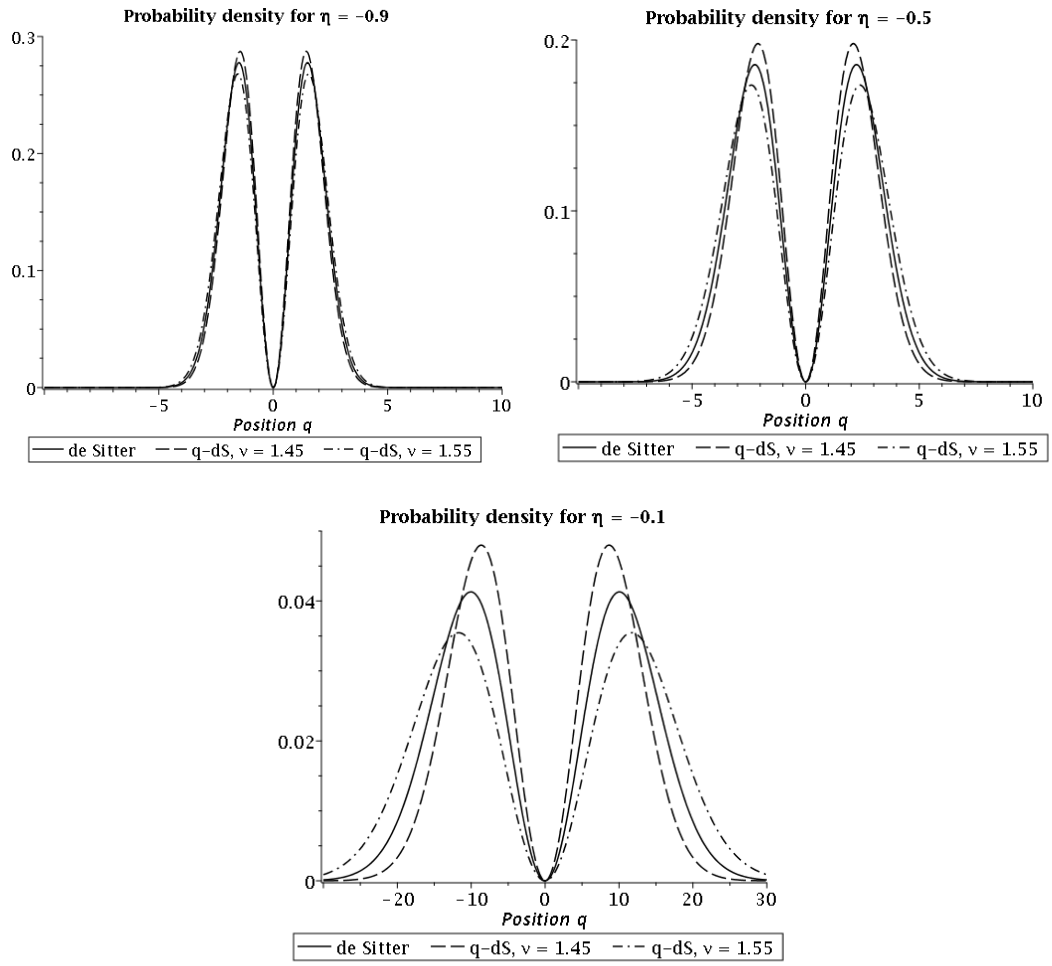


Figure 4: Modulus square of the first excited state eigenfunction of I_{LR} depending on position q for fixed conformal times η. The line shapes indicate the background universe (de Sitter or quasi-de Sitter)

4 Summary and Outlook

We analysed the eigenstates of the Lewis-Riesenfeld invariant for a quasi-de Sitter universe in the position representation and compared our solutions to the ones for de Sitter universes derived in [2]. By using the relation of the Lewis-Riesenfeld invariant to a time-independent harmonic oscillator which was implemented by a Bogoliubov transformation, we found the general spectrum of this invariant and calculated it explicitly in the position representation for the ground and first excited state. For the former, the resulting state has an oscillating probability distribution damped by a Gaussian in the position. For the latter, the probability distribution is modified by a prefactor containing position and (conformal) time. Our solution involves $\xi(\eta)$, which is the solution to the Ermakov equation. To be able to provide a fully explicit representation of the eigenstates, we had to find an expression for $\xi(\eta)$. The used method to construct $\xi(\eta)$ required a solution of the Mukhanov-Sasaki equation for a specific universe. This equation turned out to have the form of a generalized Bessel differential equation for a quasi-de Sitter universe. We derived solutions for it in terms of all three kinds of Bessel functions and showed their equivalence. These results are in accordance to the ones found in literature for a de Sitter universe by performing the limit of vanishing slow-roll parameters. As a next step we constructed $\xi(\eta)$ and used it to get an explicit expression for the lowest two eigenstates of the Lewis-Riesenfeld invariant in position representation. Graphical plots of these two states were then analysed with the result that both modulus square are almost constant in time except for conformal times close to zero, where they spread and decrease in position. With our results it is now possible to construct the entire spectrum of the invariant in position representation for a (quasi-) de Sitter universe, using the introduced ladder operators for the invariant along with the Bogoliubov transformation and the derived result for the Ermakov equation.

To continue the investigation of the perturbations and their influence on the inflaton, one could derive an expression for the general n-th eigenstate of the Lewis-Riesenfeld invariant I_{LR} in the position representation, starting with equation (70). This expression evaluated in the position representation has a similar form to the Hermite polynomials used for the eigenstates of the time-independent harmonic oscillator in that representation. But as the terms involved here do contain complex parts, the resulting expression differs from the Hermite polynomials by a sign factor, so the well-known Hermite polynomials can not be used to find a closed expression for a generic excited state of the Lewis-Riesenfeld invariant.
Referring to [2], another step would be to construct the general time evolution operator $\hat{U}(\eta_i, \eta)$ based on the Bogoliubov transformation. This was already achieved in [2]:

$$\hat{U}(\eta_i, \eta) = exp\left(-i\, \hat{I}_{LR} \int_{\eta_i}^{\eta} \frac{d\tilde{\eta}}{\xi^2(\tilde{\eta})}\right) \hat{\Gamma}^{\dagger}_{\xi(\eta)}\, \hat{\Gamma}_{\xi(\eta_i)} \; . \tag{139}$$

The problem here is the phase factor that includes an integral: The integration for de Sitter with

$$\xi(\eta) = \sqrt{1 + \frac{1}{(k\eta)^2}} \tag{140}$$

and $\eta_i = -\infty$ yields:

$$\int_{\eta_i}^{\eta} \frac{d\tilde{\eta}}{\xi^2(\tilde{\eta})} = \left[\tilde{\eta} - \frac{\arctan(k\tilde{\eta})}{k}\right]_{-\infty}^{\eta} \ . \tag{141}$$

For the lower boundary the first summand diverges. This divergence cannot be lifted by the second summand or by the upper boundary, which lies in the interval $\eta \in (-\infty, 0)$. For quasi-de Sitter we found two equivalent solutions for $\xi(\eta)$ where the one in terms of Bessel and Neumann functions was of the form

$$\xi(\eta) = \sqrt{-\frac{\pi k\eta}{2} \ \left(J_\nu^2(-k\eta) + N_\nu^2(-k\eta)\right)} \ . \tag{142}$$

We derived those two solutions explicitly by requiring well-definedness of the limit for conformal time to past infinity and by the relation for the Wronskian determinant (see (86)). From these two conditions, we got as an outcome

$$\lim_{\eta \to -\infty} \xi(\eta) = 1 \ . \tag{143}$$

Thus, the integrand in (139) does not vanish for $\eta_i \to -\infty$ but yields 1 and therefore a finite area greater than zero. So, also for a quasi-de Sitter spacetime the evolution operator does not converge for a choice of $\eta \in (-\infty, 0)$. To find a reasonable expression and be able to calculate time propagation within this formalism, one has to find a suitable regularisation for the integral in (139).

References

[1] Planck Collaboration. Planck 2015 results - XVI. Isotropy and statistics of the CMB. *A&A*, 594, 2016. URL https://doi.org/10.1051/0004-6361/201526681.

[2] Michael Kobler. Dynamical Properties of the Mukhanov-Sasaki Hamiltonian. Master's thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg, Institut für Quantengravitation, 2 2018.

[3] Eckhard Rebhan. *Theoretische Physik: Relativitätstheorie und Kosmologie*. Spektrum Akademischer Verlag, 2012. ISBN 9783827423146.

[4] Viatcheslaw Mukhanov. *Physical Foundations of Cosmology*. Cambridge University Press, 2005. ISBN 9780511136795.

[5] John G. Hartley and John R. Ray. Coherent states for the time-dependent harmonic oscillator. *Phys. Rev. D*, 25, 1982. URL https://link.aps.org/doi/10.1103/PhysRevD.25.382.

[6] K. Andriopoulos and P.G.L. Leach. The Ermakov equation: A commentary. *Applicable Analysis and Discrete Mathematics*, 2, 10 2008.

[7] Wolfgang Nolting. *Grundkurs Theoretische Physik 5/2: Quantenmechanik - Methoden und Anwendungen*. Springer Spektrum, 2015. ISBN 9783662442302.

[8] Eric W. Weisstein. Bessel function of the first kind. *From MathWorld–A Wolfram Web Resource.*, . URL http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html. [accessed 12-July-2018].

[9] Eric W. Weisstein. Bessel function of the second kind. *From MathWorld–A Wolfram Web Resource.*, . URL http://mathworld.wolfram.com/BesselFunctionoftheSecondKind.html. [accessed 12-July-2018].

[10] Nikos Drakos. Bessel functions with order half an odd integer. URL http://www.nbi.dk/~polesen/borel/node16.html. [accessed 12-July-2018].